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The Explicit Eigenvalues and Eigenfunctions of the Generalized Neumann Kernel in Annular Domains

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ABSTRACT

This paper presents the explicit forms of the eigenvalues of the generalized Neumann kernel in annular domains and the explicit forms of the corresponding eigenfunctions. It follows from these explicit forms that, except for -1, the eigenvalues are in the unit disk and clustered around 0.

Keywords: Generalized Neumann kernel, eigenvalues, eigenfunctions, Fourier series.

1. Introduction

The boundary integral equation with the generalized Neumann kernel is considerably new integral equation where the first few papers related to this integral equation are Murid and Nasser (2003), Murid et al. (2002), Wegmann et al. (2005), Wegmann and Nasser (2008). Nowadays, the integral equation has many applications to several problems in applied mathematics. Indeed, the integral equation has been successfully used to compute the conformal mapping onto more than 40 canonical domains, see Nasser (2009a,b, 2011, 2013, 2015a), Nasser and Al-Shihri (2013), Nasser et al. (2016), and to solve several boundary value problems such as the Dirichlet problem, the Neumann problem, and the mixed boundary value problem, see Crowdy et al. (2016), Nasser et al. (2012, 2011), Wegmann et al. (2005), Wegmann and Nasser (2008). Further, the integral equation has been used to compute ideal fluid flow in domains with complex boundaries, see Nasser (2015b), Nasser and Green (2018), Nasser et al. (2015).

The boundary integral equation with the generalized Neumann kernel can be solved accurately for domains with close-to-touching boundaries, non-convex boundaries, piecewise smooth boundaries, and for domains of high connectivity, see Nasser (2015b), Nasser and Al-Shihri (2013), Nasser and Green (2018), Nasser et al. (2015). For multiply connected domains of connectivity m, discretizing the boundary integral equation by the Nyström method with the trapezoidal rule yields dense and nonsymmetric $mn \times mn$ linear systems where n is the number of nodes in the discretization of each boundary component. The discretized linear system is solved by the generalized minimal residual (GMRES) method, Saad and Schultz (1986), where the matrix-vector products is calculated using the Fast Multipole Method (FMM), see Greengard and Rokhlin (1987). As demonstrated by several numerical examples in several papers, Nasser (2015a,b), Nasser and Al-Shihri (2013), Nasser and Green (2018), Nasser et al. (2016, 2015), the number of GMRES iterations required for obtaining a very good approximation of the exact solution is virtually independent of the given domain and the number of nodes n in the discretization of its boundary. As was mentioned in Nasser et al. (2016), the very fast convergence of GMRES could be due to the strong clustering around 1 for the eigenvalues of the matrix of the discretized linear system. In fact the eigenvalues of the integral equation with the generalized Neumann kernel have been studied numerically in Nasser (2015a,b), Nasser and Al-Shihri (2013), Nasser et al. (2011). The numerical results presented in these papers illustrated that, except for -1, the eigenvalues of the generalized Neumann kernel are in the unit disk and clustered around 0 (and hence the eigenvalues of the matrix of the discretized linear system, except for 2, are in the disk centered at 1 with radius 1 and these eigenvalues are clustered around 1). However, until now, there has been no analytical study on such property of the eigenvalues of the generalized Neumann kernel.

This paper presents an attempt to study analytically the properties of the eigenvalues of the generalized Neumann kernel. However, we shall limit our study to only annular domains. We present the explicit forms of these eigenvalues and their corresponding eigenfunctions. It follows from these explicit forms that, except for -1, the the eigenvalues of the generalized Neumann kernel are indeed in the unit disk and clustered around 0. Thus the analytical results of this paper confirm the numerical results presented in Nasser (2015a,b), Nasser and Al-Shihri (2013), Nasser et al. (2011) for general multiply connected domains.

2. The generalized Neumann kernel

Suppose that G is the annulus $\{z \in \mathbb{C} : q < |z| < 1\}$ with the boundary $\Gamma := \partial G = \Gamma_0 \cup \Gamma_1$, where Γ_0 is the unit circle parametrized by $\eta_0(t) = e^{\mathrm{i}t}$, $t \in J_0 = [0, 2\pi]$, and Γ_1 is the circle $|z| = q$ parametrized by $\eta_1(t) = qe^{-\mathrm{i}t}$, $t \in J_1 = [0, 2\pi]$. Let J be the disjoint union of the two intervals J_0, J_1 which is defined by

$$J = J_0 \bigsqcup J_1 = \{(t, 0) \; : \; t \in J_0\} \bigcup \{(t, 1) \; : \; t \in J_1\}.$$

The elements of J are order pairs (t, j) where j is an auxiliary index indicating which of the intervals the point t lies in. Thus, the parametrization of the whole boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ is defined as the complex function η defined on J by

$$\eta(t, j) = \eta_j(t), \quad t \in J_j, \quad j = 0, 1. \tag{1}$$

We assume that for a given t that the auxiliary index j is known, so we replace the pair (t, j) in the left-hand side of (1) by t, i.e., for a given point $t \in J$, we always know the interval J_j that contains t. The function η in (1) is thus simply written as

$$\eta(t) := \left\{ \begin{array}{ll} \eta_0(t) = e^{\mathrm{i}t}, & t \in J_0 = [0, 2\pi], \\ \eta_1(t) = qe^{-\mathrm{i}t}, & t \in J_1 = [0, 2\pi]. \end{array} \right. \tag{2}$$

Further, for any complex or real valued function ϕ defined on Γ, we shall denote the restriction of the function ϕ to the boundary Γ_j by ϕ_j for each $j = 0, 1$. In view of the parametrization η of the boundary Γ, a function ϕ defined on Γ can be interpreted via $\hat{\phi}(t) := \phi(\eta(t))$ as a 2π-periodic function in the parameter $t \in J$; and vice versa. So, in this paper, for any given complex or real valued function ϕ defined on Γ, we shall not distinguish between $\phi(t)$ and $\phi(\eta(t))$.

Let A be the complex function defined on Γ by

$$A(t) = e^{\mathrm{i}\nu(t)}(\eta(t) - p), \tag{3}$$

where p is a fixed point in G and $\nu(t)$ is the piecewise constant function

$$\nu(t) := \begin{cases} \hat{\theta}, & t \in J_0, \\ \theta, & t \in J_1, \end{cases} \tag{4}$$

with real constants $\hat{\theta}$ and θ. Without lost of generality, we assume that p is a positive real number and $\hat{\theta} = 0$, i.e., the function A is given by

$$A(t) := \begin{cases} A_0(t) = e^{\mathrm{i}t} - p, & t \in J_0, \\ A_1(t) = e^{\mathrm{i}\theta}(qe^{-\mathrm{i}t} - p), & t \in J_1. \end{cases} \tag{5}$$

The generalized Neumann kernel $N(s,t)$ formed with A and η is defined by

$$N(s,t) = \frac{1}{\pi} \operatorname{Im}\left(\frac{A(s)}{A(t)} \frac{\dot{\eta}(t)}{\eta(t) - \eta(s)} \right), \tag{6}$$

see Murid and Nasser (2003), Wegmann et al. (2005), Wegmann and Nasser (2008). Let H denote the space of all real Hölder continuous functions on the boundary Γ. We define the integral operator with the kernel $N(s,t)$ on the space H by

$$\mathbf{N}\mu := \int_J N(s,t)\mu(t)dt. \tag{7}$$

The kernel $N(s,t)$ is continuous and hence $\mathbf{N}$ is compact. For more details, we refer the reader to Nasser (2015b), Wegmann et al. (2005), Wegmann and Nasser (2008).

The objective of the this paper is to find the explicit forms of the eigenvalues and eigenfunctions of the generalized Neumann kernel $N(s,t)$, i.e., to find the explicit forms of the values of the constant λ such that the integral equation

$$\mathbf{N}\phi = \lambda\phi \tag{8}$$

has a nontrivial solution $\phi(t)$. Such a function ϕ is called the eigenfunction corresponding to the eigenvalue λ. We shall also find the explicit forms of the eigenfunctions $\phi(t)$.

For $\lambda = \pm 1$, the dimension of the null space $\mathrm{Null}(\lambda\mathbf{I} - \mathbf{N})$ has been studied in Wegmann and Nasser (2008). For the function A defined in (5), we have

$$\dim(\mathrm{Null}(\mathbf{I} - \mathbf{N})) = 0, \qquad \dim(\mathrm{Null}(\mathbf{I} + \mathbf{N})) = 2, \tag{9}$$

i.e., $\lambda = 1$ is not an eigenvalue of $N(s,t)$ and $\lambda = -1$ is an eigenvalue of $N(s,t)$ with two linearly independent eigenfunctions, see Nasser (2009a). Further, the explicit forms of the eigenfunctions of $N(s,t)$ corresponding to the eigenvalue $\lambda = -1$ are known as in the following theorem which has been proved in Nasser (2009a) for the above function A with $\theta = 0$. However, the proof is also valid for $\theta \neq 0$. An alternative proof will be given in Section 5 below.

Theorem 2.1. *The null space* $\mathrm{Null}(\mathbf{I} + \mathbf{N})$ *is given by*

$$\mathrm{Null}(\mathbf{I} + \mathbf{N}) = \mathrm{span}\{\chi, \hat{\chi}\} \tag{10}$$

where

$$\chi(t) = \begin{cases} 1, & t \in J_0, \\ 0, & t \in J_1, \end{cases} \quad \text{and} \quad \hat{\chi}(t) = \begin{cases} 0, & t \in J_0, \\ 1, & t \in J_1. \end{cases} \tag{11}$$

In this paper, we present the explicit forms of the all eigenvalues of the generalized Neumann kernel $N(s,t)$ and the explicit forms of their corresponding eigenfunctions for the above prescribed annular domains. We will need the following theorem from Wegmann and Nasser (2008).

Theorem 2.2. *If $\lambda \neq -1$ is an eigenvalue of $N(s,t)$, then $-\lambda$ is also an eigenvalue of $N(s,t)$.*

3. The kernels

For $i, j = 1, 2$, we define the restriction of the kernel $N(s,t)$ to $J_i \times J_j$ by $N_{ij}(s,t)$ where $s \in J_i$ and $t \in J_j$, i.e.,

$$N_{ij}(s,t) = \frac{1}{\pi} \mathrm{Im} \left(\frac{A_i(s)}{A_j(t)} \frac{\dot{\eta}_j(t)}{\eta_j(t) - \eta_i(s)} \right). \tag{12}$$

We shall write the kernels $N_{ij}(s,t)$ in a convenient form for using Fourier series. Based on (2) and (5) and since $0<p<1$, we have for $i=j=0$,

$$\begin{aligned}
N_{00}(s,t) &= \frac{1}{\pi}\,\mathrm{Im}\left(\frac{e^{\mathrm{i}s}-p}{e^{\mathrm{i}t}-p}\frac{\mathrm{i}e^{\mathrm{i}t}}{e^{\mathrm{i}t}-e^{\mathrm{i}s}}\right)\\
&= \frac{1}{\pi}\,\mathrm{Im}\left(\frac{\mathrm{i}e^{\mathrm{i}t}}{e^{\mathrm{i}t}-e^{\mathrm{i}s}}-\frac{\mathrm{i}e^{\mathrm{i}t}}{e^{\mathrm{i}t}-p}\right)\\
&= \mathrm{Re}\left(\frac{1}{\pi}\frac{e^{-\mathrm{i}(s-t)/2}}{e^{-\mathrm{i}(s-t)/2}-e^{\mathrm{i}(s-t)/2}}-\frac{1}{\pi}\frac{1}{1-pe^{-\mathrm{i}t}}\right)\\
&= \mathrm{Re}\left(\frac{1}{\pi}\frac{\cos\frac{s-t}{2}-\mathrm{i}\sin\frac{s-t}{2}}{-2\mathrm{i}\sin\frac{s-t}{2}}-\frac{1}{\pi}\sum_{k=0}^{\infty}p^k e^{-\mathrm{i}kt}\right)\\
&= \frac{1}{2\pi}-\frac{1}{\pi}\sum_{k=0}^{\infty}p^k\cos kt.
\end{aligned}$$

Hence, we have

$$N_{00}(s,t) = -\frac{1}{2\pi}-\frac{1}{\pi}\sum_{k=1}^{\infty}p^k\cos kt. \tag{13}$$

For $i=0$ and $j=1$, we have

$$\begin{aligned}
N_{01}(s,t) &= \frac{1}{\pi}\,\mathrm{Im}\left(\frac{e^{\mathrm{i}s}-p}{e^{\mathrm{i}\theta}(qe^{-\mathrm{i}t}-p)}\frac{-\mathrm{i}qe^{\mathrm{i}t}}{qe^{-\mathrm{i}t}-e^{\mathrm{i}s}}\right)\\
&= \frac{1}{\pi}\,\mathrm{Im}\left(e^{-\mathrm{i}\theta}\frac{-\mathrm{i}qe^{-\mathrm{i}t}}{qe^{-\mathrm{i}t}-e^{\mathrm{i}s}}-e^{-\mathrm{i}\theta}\frac{-\mathrm{i}qe^{-\mathrm{i}t}}{qe^{-\mathrm{i}t}-p}\right)\\
&= \frac{1}{\pi}\,\mathrm{Im}\left(e^{-\mathrm{i}\theta}\frac{\mathrm{i}qe^{-\mathrm{i}(t+s)}}{1-qe^{-\mathrm{i}(t+s)}}-e^{-\mathrm{i}\theta}\frac{\mathrm{i}(q/p)e^{-\mathrm{i}t}}{1-(q/p)e^{-\mathrm{i}t}}\right)\\
&= \mathrm{Re}\left(e^{-\mathrm{i}\theta}\frac{1}{\pi}\sum_{k=1}^{\infty}q^k e^{-\mathrm{i}k(t+s)}-e^{-\mathrm{i}\theta}\frac{1}{\pi}\sum_{k=1}^{\infty}(q/p)^k e^{-\mathrm{i}kt}\right),
\end{aligned}$$

since $0<q<1$ and $0<q/p<1$. Hence, we have

$$\begin{aligned}
N_{01}(s,t) &= \cos\theta\left[\frac{1}{\pi}\sum_{k=1}^{\infty}q^k\cos(k(t+s))-\frac{1}{\pi}\sum_{k=1}^{\infty}(q/p)^k\cos(kt)\right]\\
&\quad -\sin\theta\left[\frac{1}{\pi}\sum_{k=1}^{\infty}q^k\sin(k(t+s))-\frac{1}{\pi}\sum_{k=1}^{\infty}(q/p)^k\sin(kt)\right]
\end{aligned} \tag{14}$$

Similarly, we can show that

$$\begin{aligned} N_{10}(s,t) &= \cos\theta\left[\frac{1}{\pi}\sum_{k=1}^{\infty} q^k\cos(k(t+s)) - \frac{1}{\pi}\sum_{k=1}^{\infty} p^k\cos(kt)\right] \\ &\quad + \sin\theta\left[\frac{1}{\pi}\sum_{k=1}^{\infty} q^k\sin(k(t+s)) - \frac{1}{\pi}\sum_{k=1}^{\infty} p^k\sin(kt)\right], \qquad (15) \end{aligned}$$

and

$$N_{11}(s,t) = -\frac{1}{2\pi} - \frac{1}{\pi}\sum_{k=1}^{\infty}(q/p)^k\cos kt. \qquad (16)$$

Lemma 3.1. *For $i,j=0,1$,*

$$\int_0^{2\pi} N_{ij}(s,t)dt = -\delta_{ij} \qquad (17a)$$

where δ_{ij} is the Kronecker delta function. Further, for $n=1,2,3,\ldots$, we have

$$\int_0^{2\pi} N_{00}(s,t)\left\{\begin{array}{c}\cos nt\\ \sin nt\end{array}\right\}dt = \left\{\begin{array}{c}-p^n\\ 0\end{array}\right\}, \qquad (17b)$$

$$\int_0^{2\pi} N_{01}(s,t)\left\{\begin{array}{c}\cos nt\\ \sin nt\end{array}\right\}dt = \left\{\begin{array}{c}\cos\theta[q^n\cos ns-(q/p)^n]-\sin\theta[q^n\sin ns]\\ \cos\theta[-q^n\sin ns]-\sin\theta[q^n\cos ns-(q/p)^n]\end{array}\right\}, \qquad (17c)$$

$$\int_0^{2\pi} N_{10}(s,t)\left\{\begin{array}{c}\cos nt\\ \sin nt\end{array}\right\}dt = \left\{\begin{array}{c}\cos\theta[q^n\cos ns-p^n]+\sin\theta[q^n\sin\theta]\\ \cos\theta[-q^n\sin ns]+\sin\theta[q^n\cos ns-p^n]\end{array}\right\}, \qquad (17d)$$

$$\int_0^{2\pi} N_{11}(s,t)\left\{\begin{array}{c}\cos nt\\ \sin nt\end{array}\right\}dt = \left\{\begin{array}{c}-(q/p)^n\\ 0\end{array}\right\}. \qquad (17e)$$

Proof. It is clear from (13) and (16) that

$$\int_0^{2\pi} N_{00}(s,t)dt = \int_0^{2\pi} N_{11}(s,t)dt = -1.$$

Further, using the identities

$$\cos(k(t+s)) = \cos kt\cos ks - \sin kt\sin ks,, \qquad (18a)$$
$$\sin(k(t+s)) = \sin kt\cos ks + \cos kt\sin ks, \qquad (18b)$$

it follows from (14) and (15)

$$\int_0^{2\pi} N_{01}(s,t)dt = \int_0^{2\pi} N_{10}(s,t)dt = 0.$$

Hence, Equation (17a) is proven.

To prove Equation (17b) for $n = 1, 2, 3, \ldots$, it follows from (13) that

$$\int_0^{2\pi} N_{00}(s,t)\cos nt\,dt = -\frac{1}{2\pi}\int_0^{2\pi}\cos nt\,dt - \frac{1}{\pi}\sum_{k=1}^{\infty} p^k \int_0^{2\pi}\cos kt\cos nt\,dt = -p^n$$

and

$$\int_0^{2\pi} N_{00}(s,t)\sin nt\,dt = -\frac{1}{2\pi}\int_0^{2\pi}\sin nt\,dt - \frac{1}{\pi}\sum_{k=1}^{\infty} p^k \int_0^{2\pi}\cos kt\sin nt\,dt = 0.$$

Similarly, to prove Equation (17c) for $n = 1, 2, 3, \ldots$, it follows from (14) and (18) that

$$\begin{aligned}\int_0^{2\pi} N_{01}(s,t)\cos nt\,dt \quad = \quad & \frac{\cos\theta}{\pi}\sum_{k=1}^{\infty} q^k \int_0^{2\pi}(\cos kt\cos ks - \sin kt\sin ks)\cos nt\,dt \\ & -\frac{\cos\theta}{\pi}\sum_{k=1}^{\infty}(q/p)^k \int_0^{2\pi}\cos(kt)\cos nt\,dt \\ & -\frac{\sin\theta}{\pi}\sum_{k=1}^{\infty} q^k \int_0^{2\pi}(\sin kt\cos ks + \cos kt\sin ks)\cos nt\,dt \\ & +\frac{\sin\theta}{\pi}\sum_{k=1}^{\infty}(q/p)^k \int_0^{2\pi}\sin(kt)\cos nt\,dt,\end{aligned}$$

which implies that

$$\int_0^{2\pi} N_{01}(s,t)\cos nt\,dt = \cos\theta q^n\cos ns - \cos\theta(q/p)^n - \sin\theta q^n\sin ns.$$

so the first part in (17c) is proven. Further, we have

$$\begin{aligned}\int_0^{2\pi} N_{01}(s,t)\sin nt\,dt \quad = \quad & \frac{\cos\theta}{\pi}\sum_{k=1}^{\infty} q^k \int_0^{2\pi}(\cos kt\cos ks - \sin kt\sin ks)\sin nt\,dt \\ & -\frac{\cos\theta}{\pi}\sum_{k=1}^{\infty}(q/p)^k \int_0^{2\pi}\cos(kt)\sin nt\,dt \\ & -\frac{\sin\theta}{\pi}\sum_{k=1}^{\infty} q^k \int_0^{2\pi}(\sin kt\cos ks + \cos kt\sin ks)\sin nt\,dt \\ & +\frac{\sin\theta}{\pi}\sum_{k=1}^{\infty}(q/p)^k \int_0^{2\pi}\sin(kt)\sin nt\,dt,\end{aligned}$$

which implies that

$$\int_0^{2\pi} N_{01}(s,t)\cos nt\,dt = -\cos\theta q^n \sin ns - \sin\theta q^n \cos ns + \sin\theta (q/p)^n,$$

and the second part in (17c) is proven.

Equations (17d) and (17e) can be proven in the same way. □

4. Eigenvalues

In this section, with the help of the Fourier series, the Formulas (17b)-(17e) will be used to find the explicit forms of the eigenvalues λ of the generalized Neumann kernel. The eigenvalues λ could be complex numbers and hence the eigenfunctions $\phi(t)$ could be complex functions in the real variable t. Suppose the function ϕ defined as

$$\phi(t) = \begin{cases} \phi_0(t), & t \in J_0, \\ \phi_1(t), & t \in J_1, \end{cases} \tag{19}$$

has the Fourier series expansion

$$\phi_i(t) = a_{i,0} + \sum_{n=1}^{\infty} a_{i,n}\cos nt + \sum_{n=1}^{\infty} b_{i,n}\sin nt, \quad t \in J_i, \tag{20}$$

with complex constants $a_{i,0}$, $a_{i.n}$ and $b_{i,n}$ for $i = 0, 1$ and $n = 1, 2, 3, \ldots$. Hence the integral equation

$$\mathbf{N}\phi = \lambda\phi$$

can be written as the following 2×2 system of integral equations

$$\int_0^{2\pi} N_{00}(s,t)\phi_0(t)dt + \int_0^{2\pi} N_{01}(s,t)\phi_1(t)dt = \lambda\phi_0(s), \tag{21}$$

$$\int_0^{2\pi} N_{10}(s,t)\phi_0(t)dt + \int_0^{2\pi} N_{11}(s,t)\phi_1(t)dt = \lambda\phi_1(s). \tag{22}$$

We will compute each integral in (21)–(22) separately. For the first integral, we have

$$\int_0^{2\pi} N_{00}(s,t)\phi_0(t)dt = \int_0^{2\pi} N_{00}(s,t)\left[a_{0,0} + \sum_{n=1}^{\infty} a_{0,n}\cos nt + \sum_{n=1}^{\infty} b_{0,n}\sin nt\right] dt$$

which, in view of (17), implies that

$$\int_0^{2\pi} N_{00}(s,t)\phi_0(t)dt = -a_{0,0} - \sum_{n=1}^{\infty} a_{0,n}p^n. \tag{23}$$

Similarly, in view of (17), we have

$$\begin{aligned}\int_0^{2\pi} N_{01}(s,t)\phi_1(t)dt &= \sum_{n=1}^{\infty}[a_{1,n}\cos\theta - b_{1,n}\sin\theta]q^n\cos ns \\ &\quad - \sum_{n=1}^{\infty}[a_{1,n}\cos\theta - b_{1,n}\sin\theta](q/p)^n \\ &\quad - \sum_{n=1}^{\infty}[a_{1,n}\sin\theta + b_{1,n}\cos\theta]q^n\sin ns,\end{aligned} \tag{24}$$

$$\begin{aligned}\int_0^{2\pi} N_{10}(s,t)\phi_0(t)dt &= \sum_{n=1}^{\infty}[a_{0,n}\cos\theta + b_{0,n}\sin\theta]q^n\cos ns \\ &\quad - \sum_{n=1}^{\infty}[a_{0,n}\cos\theta + b_{0,n}\sin\theta]p^n \\ &\quad + \sum_{n=1}^{\infty}[a_{0,n}\sin\theta - b_{0,n}\cos\theta]q^n\sin ns,\end{aligned} \tag{25}$$

and

$$\int_0^{2\pi} N_{11}(s,t)\phi_1(t)dt = -a_{1,0} - \sum_{n=1}^{\infty} a_{1,n}(q/p)^n. \tag{26}$$

Consequently, by substituting (23) and (24) into (21), we obtain

$$\begin{aligned}&-a_{0,0} - \sum_{n=1}^{\infty} a_{0,n}p^n + \sum_{n=1}^{\infty}[a_{1,n}\cos\theta - b_{1,n}\sin\theta]q^n\cos ns \\ &- \sum_{n=1}^{\infty}[a_{1,n}\cos\theta - b_{1,n}\sin\theta](q/p)^n - \sum_{n=1}^{\infty}[a_{1,n}\sin\theta + b_{1,n}\cos\theta]q^n\sin ns \\ &= \lambda a_{0,0} + \sum_{n=1}^{\infty}\lambda a_{0,n}\cos ns + \sum_{n=1}^{\infty}\lambda b_{0,n}\sin ns.\end{aligned}$$

Similarly, substituting (25) and (26) into (22) yields

$$\begin{aligned}&\sum_{n=1}^{\infty}[a_{0,n}\cos\theta + b_{0,n}\sin\theta]q^n\cos ns - \sum_{n=1}^{\infty}[a_{0,n}\cos\theta + b_{0,n}\sin\theta]p^n \\ &+ \sum_{n=1}^{\infty}[a_{0,n}\sin\theta - b_{0,n}\cos\theta]q^n\sin ns - a_{1,0} - \sum_{n=1}^{\infty} a_{1,n}(q/p)^n \\ &= \lambda a_{1,0} + \sum_{n=1}^{\infty}\lambda a_{1,n}\cos ns + \sum_{n=1}^{\infty}\lambda b_{1,n}\sin ns.\end{aligned}$$

Then, by equating similar terms in both sides of the previous two equations, the function ϕ in (19) is an eigenfunction of $N(s,t)$ corresponding to the eigenvalue λ if the constants $a_{i,0}$, $a_{i.n}$ and $b_{i,n}$ for $i = 0, 1$ and $n = 1, 2, 3, \ldots$, satisfy the following equations:

$$\lambda a_{0,0} = -a_{0,0} - \sum_{n=1}^{\infty} a_{0,n} p^n - \sum_{n=1}^{\infty} [a_{1,n}\cos\theta - b_{1,n}\sin\theta](q/p)^n, \quad (27a)$$

$$\lambda a_{0,n} = [a_{1,n}\cos\theta - b_{1,n}\sin\theta]q^n, \quad n = 1, 2, 3, \ldots, \quad (27b)$$

$$\lambda b_{0,n} = -[a_{1,n}\sin\theta + b_{1,n}\cos\theta]q^n, \quad n = 1, 2, 3, \ldots, \quad (27c)$$

$$\lambda a_{1,0} = -a_{1,0} - \sum_{n=1}^{\infty} a_{1,n}(q/p)^n - \sum_{n=1}^{\infty} [a_{0,n}\cos\theta + b_{0,n}\sin\theta]p^n, \quad (27d)$$

$$\lambda a_{1,n} = [a_{0,n}\cos\theta + b_{0,n}\sin\theta]q^n, \quad n = 1, 2, 3, \ldots, \quad (27e)$$

$$\lambda b_{1,n} = [a_{0,n}\sin\theta - b_{0,n}\cos\theta]q^n, \quad n = 1, 2, 3, \ldots. \quad (27f)$$

Theorem 4.1. *For $m = 1, 2, 3, \ldots$,*

$$\lambda = q^m e^{\mathrm{i}\theta}$$

are eigenvalues of the generalized Neumann kernel $N(s,t)$.

Proof. We shall show that $\lambda = q^m e^{\mathrm{i}\theta}$ by showing that Equations (27a)–(27f) have a nontrivial solution for such values of λ.

For a fixed m, $m = 1, 2, 3, \ldots$, choose:

$$\begin{aligned} a_{0,m} = 1, \quad a_{1,m} = 1, \quad & b_{0,m} = \mathrm{i}, \quad b_{1,m} = -\mathrm{i} \\ a_{0,n} = a_{1,n} = 0 \quad & \text{for } n = 1, 2, 3, \ldots \quad \text{and } n \neq m, \\ b_{0,n} = b_{1,n} = 0 \quad & \text{for } n = 1, 2, 3, \ldots. \end{aligned} \quad (28)$$

It is clear that Equations (27b), (27c), (27e), and (27f) are satisfied with such chosen values. For Equations (27a) and (27c) to be satisfied, we substitute the values (28) into (27a) to obtain

$$q^m e^{\mathrm{i}\theta} a_{0,0} = -a_{0,0} - p^m - (\cos\theta + \mathrm{i}\sin\theta)(q/p)^m.$$

Hence, Equation (27a) is satisfied if we choose

$$a_{0,0} = -\frac{p^{2m} + q^m e^{\mathrm{i}\theta}}{p^m(1 + q^m e^{\mathrm{i}\theta})}.$$

Also, we substitute these chosen values (28) into (27d) to obtain

$$q^m e^{\mathrm{i}\theta} a_{1,0} = -a_{1,0} - (q/p)^m - (\cos\theta + \mathrm{i}\sin\theta)p^m.$$

Then, Equation (27d) is satisfied if we choose

$$a_{1,0} = -\frac{p^{2m}e^{\mathrm{i}\theta} + q^m}{p^m(1 + q^m e^{\mathrm{i}\theta})}.$$

Thus, we have proved that, for $m = 1, 2, 3, \ldots$, a nontrivial solution to Equations (27) always exists for $\lambda = q^m e^{\mathrm{i}\theta}$ and hence $\lambda = q^m e^{\mathrm{i}\theta}$ are eigenvalues of the generalized Neumann kernel. □

Corollary 4.1. *For $m = 1, 2, 3, \ldots$,*

$$\pm q^m e^{\pm \mathrm{i}\theta}$$

are eigenvalues of the generalized Neumann kernel $N(s,t)$.

Proof. It follows from the previous theorem that, for $m = 1, 2, 3, \ldots$, $\lambda = q^m e^{\mathrm{i}\theta}$ are eigenvalues of $N(s,t)$. Since the generalized Neumann kernel $N(s,t)$ is a real kernel, then $\overline{\lambda} = q^m e^{-\mathrm{i}\theta}$ are also eigenvalues of $N(s,t)$. Finally, Theorem 2.2 implies that $-\lambda = -q^m e^{\mathrm{i}\theta}$ and $-\overline{\lambda} = -q^m e^{-\mathrm{i}\theta}$ are also eigenvalues of $N(s,t)$. □

Note that (27b) and (27c) can be written as

$$\lambda \begin{pmatrix} a_{0,n} \\ b_{0,n} \end{pmatrix} = q^n \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} a_{1,n} \\ b_{1,n} \end{pmatrix}. \tag{29}$$

Similarly, (27e) and (27f) can be written as

$$\lambda \begin{pmatrix} a_{1,n} \\ b_{1,n} \end{pmatrix} = q^n \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} a_{0,n} \\ b_{0,n} \end{pmatrix}. \tag{30}$$

Thus, substituting (30) into (29) yields

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} a_{0,n} \\ b_{0,n} \end{pmatrix} = \lambda^2 q^{-2n} \begin{pmatrix} a_{0,n} \\ b_{0,n} \end{pmatrix}. \tag{31}$$

Similarly, by substituting (29) into (30), we obatain

$$\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} a_{1,n} \\ b_{1,n} \end{pmatrix} = \lambda^2 q^{-2n} \begin{pmatrix} a_{1,n} \\ b_{1,n} \end{pmatrix}. \tag{32}$$

Theorem 4.2. *If λ is an eigenvalues of the generalized Neumann kernel $N(s,t)$ other than $\pm q^n e^{\pm \mathrm{i}\theta}$ for $n = 1, 2, 3, \ldots$, then $\lambda = -1$.*

Proof. Assume that an eigenvalue λ of $N(s,t)$ exists such that $\lambda \neq \pm q^n e^{\pm i\theta}$ for all $n = 1, 2, 3, \ldots$. Note that the eigenvalues of the 2×2 matrices in (31) and (32) are $e^{\pm 2i\theta}$. Since $\lambda \neq \pm q^n e^{\pm i\theta}$ for all $n = 1, 2, 3, \ldots$, it follows that $\lambda^2 q^{-2n}$ is not an eigenvalue for both matrices in (31) and (32). Hence, both Equations (31) and (32) have only the zero solution

$$a_{0,n} = b_{0,n} = a_{1,n} = b_{1,n} = 0 \tag{33}$$

for all $n = 1, 2, 3, \ldots$. Hence, it follows from (27a) from (27d) that

$$(\lambda + 1)a_{0,0} = 0 \quad \text{and} \quad (\lambda + 1)a_{1,0} = 0.$$

Thus, we have $\lambda = -1$ (otherwise, we will have $a_{0,0} = 0$ and $a_{1,0} = 0$ and hence (20) and (19) implies that the function ϕ is identically zero which contradict the assumption that λ is an eigenvalue of $N(s,t)$ and ϕ is its corresponding eigenfunction). □

Corollary 4.2. *The spectrum of* $\mathbf{N}$ *is*

$$\sigma(\mathbf{N}) = \{0, -1, \pm q^m e^{\pm i\theta} \ \text{ for } \ m = 1, 2, 3, \ldots\}. \tag{34}$$

Proof. Since $\mathbf{N}$ is a compact linear operator on an infinite dimensional normed space, then $0 \in \sigma(\mathbf{N})$, see Kress (2014). The proof then follows from Corollary 4.1 and Theorem 4.2. □

Corollary 4.3. *The boundary integral operator* $\mathbf{N}$ *has spectral radius* $\rho(\mathbf{N}) = 1$.

Proof. The proof follows from Corollary 4.2 since $0 < q < 1$. □

Remark 4.1. Since the spectral radius $\rho(\mathbf{N}) = 1$, the integral equation

$$\mu - \mathbf{N}\mu = \gamma$$

can be solved approximately by successive iterations with relaxation method

$$\mu_{k+1} = (1 - \omega)\mu_k + \omega \mathbf{N}\mu_k + \gamma, \quad k = 0, 1, 2, \ldots,$$

where $0 < \omega < 1$. The approximations μ_k converges uniformly for all μ_0 and all $\gamma \in H$ to the unique solution μ of the integral equation (see González and Kress (1977), Kress (2014), Kythe (2012), Kythe and Puri (2011) for more details).

5. Eigenfunctions

From Theorems 2.2, 4.1, and 4.2, the eigenvalues of the generalized Neumann kernel are $\lambda = -1$ and $\lambda = \pm q^m e^{\pm i\theta}$ for all $m = 1, 2, 3, \ldots$. The explicit form of the eigenfunctions corresponding to the eigenvalue $\lambda = -1$ are given in Theorem 2.1. Here, we give an alternative proof of Theorem 2.1 using the above notations.

Proof of Theorem 2.1. Let the eigenfunction corresponding to the eigenvalue $\lambda = -1$ be given by

$$\phi(t) = \begin{cases} \phi_0(t), & t \in J_0, \\ \phi_1(t), & t \in J_1, \end{cases}$$

with the representation (20). Then it follows from (31) and (32) that

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} a_{0,n} \\ b_{0,n} \end{pmatrix} = q^{-2n} \begin{pmatrix} a_{0,n} \\ b_{0,n} \end{pmatrix} \tag{35}$$

and

$$\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} a_{1,n} \\ b_{1,n} \end{pmatrix} = q^{-2n} \begin{pmatrix} a_{1,n} \\ b_{1,n} \end{pmatrix} \tag{36}$$

for all $n = 1, 2, 3, \ldots$. Since the eigenvalues of the 2×2 matrices in (35) and (36) are $e^{\pm 2i\theta}$, the systems (35) and (36) have only the trivial solution, i.e.,

$$a_{0,n} = b_{0,n} = a_{1,n} = b_{1,n} = 0 \quad \text{for all} \quad n = 1, 2, 3, \ldots. \tag{37}$$

Consequently, Equations (27) are satisfied for arbitrary constant $a_{0,0}$ and $a_{1,0}$. Hence, it follows from the representation (20) that

$$\phi(t) = \begin{cases} a_{0,0}, & t \in J_0, \\ a_{1,0}, & t \in J_1, \end{cases}$$

where $a_{0,0}$ and $a_{1,0}$ are arbitrary constants. Then, by the definition of the functions $\chi(t)$ and $\hat{\chi}(t)$, we have

$$\phi(t) = a_{0,0}\chi(t) + a_{1,0}\hat{\chi}(t),$$

which implies (10). □

In the rest of this section, we assume that $\lambda \neq -1$. We shall present the explicit form the eigenfunctions corresponding to the other eigenvalues $\lambda = \pm q^m e^{\pm i\theta}$. We start with the following lemma.

Lemma 5.1. *For a fixed m, $m = 1, 2, 3, \dots$, the eigenfunction corresponding to the eigenvalue*

$$\lambda = \pm q^m e^{\pm i\theta},$$

is given by

$$\phi(t) = \begin{cases} \phi_0(t), & t \in J_0, \\ \phi_1(t), & t \in J_1. \end{cases}$$

where ϕ_i has the representation

$$\phi_i(t) = a_{i,0} + a_{i,m} \cos mt + b_{i,m} \sin mt, \quad t \in J_i, \quad i = 0, 1, \tag{38}$$

and the six constants $a_{0,0}$, $a_{0,m}$, $b_{0,m}$, $a_{1,0}$, $a_{1,m}$, and $b_{1,m}$ satisfy the equations

$$\lambda a_{0,0} = -a_{0,0} - a_{0,m} p^m - [a_{1,m} \cos\theta - b_{1,m} \sin\theta](q/p)^m, \tag{39a}$$
$$\lambda a_{0,m} = [a_{1,m} \cos\theta - b_{1,m} \sin\theta] q^m, \tag{39b}$$
$$\lambda b_{0,m} = -[a_{1,m} \sin\theta + b_{1,m} \cos\theta] q^m, \tag{39c}$$
$$\lambda a_{1,0} = -a_{1,0} - a_{1,m}(q/p)^m - [a_{0,m} \cos\theta + b_{0,m} \sin\theta] p^m, \tag{39d}$$
$$\lambda a_{1,m} = [a_{0,m} \cos\theta + b_{0,m} \sin\theta] q^m, \tag{39e}$$
$$\lambda b_{1,m} = [a_{0,m} \sin\theta - b_{0,m} \cos\theta] q^m. \tag{39f}$$

Proof. For a fixed m, $m = 1, 2, 3, \dots$, and for the eigenvalue

$$\lambda = \pm q^m e^{\pm i\theta},$$

let

$$\phi(t) = \begin{cases} \phi_0(t), & t \in J_0, \\ \phi_1(t), & t \in J_1. \end{cases}$$

be its the corresponding eigenfunction with the general representation (20). Then it follows from (31) and (32) that

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} a_{0,n} \\ b_{0,n} \end{pmatrix} = q^{2m} e^{\pm i2\theta} q^{-2n} \begin{pmatrix} a_{0,n} \\ b_{0,n} \end{pmatrix}, \tag{40}$$

and

$$\begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} a_{1,n} \\ b_{1,n} \end{pmatrix} = q^{2m} e^{\pm i2\theta} q^{-2n} \begin{pmatrix} a_{1,n} \\ b_{1,n} \end{pmatrix} \tag{41}$$

for all $n = 1, 2, 3, \dots$. Since the eigenvalues of the 2×2 matrices in (40) and (41) are $e^{\pm 2i\theta}$, the systems (40) and (41) have only the trivial solution for $n \neq m$, i.e.,

$$a_{0,n} = b_{0,n} = a_{1,n} = b_{1,n} = 0 \quad \text{for all} \quad n = 1, 2, 3, \dots \quad \text{with} \quad n \neq m. \tag{42}$$

Hence, the representation (20) reduces to the representation (38) and Equations (27) reduce to Equations (39). □

It follows from the representation (38) that the function ϕ contains six constants (in general complex), $a_{0,0}$, $a_{1,0}$, $a_{0,m}$, $a_{1,m}$, $b_{0,m}$, and $b_{1,m}$. For the function ϕ to be an eigenfunction of $N(s,t)$ corresponding to the eigenvalue $\lambda = \pm q^m e^{\pm i\theta}$, these six constants must satisfy Equations (39). It is clear that the eigenvalues $\pm q^m e^{\pm i\theta}$ are real if $\sin\theta = 0$ and are pure imaginary if $\cos\theta = 0$. If both $\sin\theta \neq 0$ and $\cos\theta \neq 0$, then we have complex eigenvalues that are neither real nor pure imaginary. So, we shall consider these three cases separately.

Remark 5.1. For $\theta = 0$, it has been proven in Nasser et al. (2011) that the eigenvalues are real in the interval $[-1, 1)$.

Remark 5.2. Note that Equations (39) can be written in matrix notations as

$$\begin{pmatrix} -1 & -p^m & 0 & 0 & -(q/p)^m\cos\theta & (q/p)^m\sin\theta \\ 0 & 0 & 0 & 0 & q^m\cos\theta & -q^m\sin\theta \\ 0 & 0 & 0 & 0 & -q^m\sin\theta & -q^m\cos\theta \\ 0 & -p^m\cos\theta & -p^m\sin\theta & -1 & -(q/p)^m & 0 \\ 0 & q^m\cos\theta & q^m\sin\theta & 0 & 0 & 0 \\ 0 & q^m\sin\theta & -q^m\cos\theta & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{0,0} \\ a_{0,m} \\ b_{0,m} \\ a_{1,0} \\ a_{1,m} \\ b_{1,m} \end{pmatrix} = \lambda \begin{pmatrix} a_{0,0} \\ a_{0,m} \\ b_{0,m} \\ a_{1,0} \\ a_{1,m} \\ b_{1,m} \end{pmatrix},$$

and hence the vector $(a_{0,0}, a_{0,m}, b_{0,m}, a_{1,0}, a_{1,m}, b_{1,m})^T$ is the eigenvector to the matrix corresponding to the eigenvalue λ. It follows from Lemma 5.1 that $q^m e^{+i\theta}$, $-q^m e^{+i\theta}$, $q^m e^{-i\theta}$, and $-q^m e^{-i\theta}$ are eigenvalues of the matrix. Further, it is clear that -1 is also an eigenvalue (repeated) of the matrix with two linearly independent eigenvectors $(1, 0, 0, 0, 0, 0, 0)^T$ and $(0, 0, 0, 1, 0, 0)^T$. We can use this matrix notations to find the eigenvectors corresponding to the eigenvalues $\pm q^m e^{\pm i\theta}$ (and hence, in view of (38), the eigenfunctions of $N(s,t)$). However, it is easy to find the eigenfunctions using Equations (39) directly rather than using the above matrix.

5.1 Case $\sin\theta = 0$

For this case the eigenvalues are real and are given by

$$\lambda = \pm q^m.$$

Theorem 5.1. *Suppose that* $\sin\theta = 0$ *and, for a fixed* m, $m = 1, 2, 3, \ldots$,

$$\lambda = q^m.$$

Then

$$\mathrm{Null}(\lambda \mathbf{I} - \mathbf{N}) = \mathrm{span}\{\varphi, \hat{\varphi}\}$$

where

$$\varphi(t) = \begin{cases} \alpha + \cos mt, & t \in J_0, \\ \alpha\cos\theta + \cos\theta\cos mt, & t \in J_1, \end{cases} \qquad \hat{\varphi}(t) = \begin{cases} \sin mt, & t \in J_0, \\ -\cos\theta \sin mt, & t \in J_1, \end{cases} \tag{43}$$

and

$$\alpha = -\frac{p^{2m} + q^m}{p^m(1+q^m)}. \tag{44}$$

Proof. Since $\sin\theta = 0$, then Equations (39) become

$$\begin{aligned}
q^m a_{0,0} &= -a_{0,0} - a_{0,m}p^m - a_{1,m}\cos\theta(q/p)^m, && \text{(45a)}\\
a_{0,m} &= a_{1,m}\cos\theta, && \text{(45b)}\\
b_{0,m} &= -b_{1,m}\cos\theta, && \text{(45c)}\\
q^m a_{1,0} &= -a_{1,0} - a_{1,m}(q/p)^m - a_{0,m}\cos\theta p^m, && \text{(45d)}\\
a_{1,m} &= a_{0,m}\cos\theta, && \text{(45e)}\\
b_{1,m} &= -b_{0,m}\cos\theta. && \text{(45f)}
\end{aligned}$$

Also, $\sin\theta = 0$ implies that $\cos^2\theta = 1$. Hence, Equations (45b) and (45e) are identical. Similarly, Equations (45c) and (45f) are identical. Substituting (45b) into (45a), we obtain

$$q^m a_{0,0} = -a_{0,0} - a_{0,m}p^m - a_{0,m}(q/p)^m$$

which implies that

$$a_{0,0} = -a_{0,m}\frac{p^{2m} + q^m}{p^m(1+q^m)}. \tag{46}$$

Similarly, substituting (45e) into (45d), we obtain

$$q^m a_{1,0} = -a_{1,0} - a_{0,m}\cos\theta(q/p)^m - a_{0,m}\cos\theta p^m$$

which implies that

$$a_{1,0} = -a_{0,m}\cos\theta\frac{p^{2m} + q^m}{p^m(1+q^m)}. \tag{47}$$

Then, in view of Lemma 5.1, the eigenfunction ϕ corresponding to the eigenvalue $\lambda = q^m$ is given by

$$\phi(t) = \begin{cases} \phi_0(t), & t \in J_0, \\ \phi_1(t), & t \in J_1. \end{cases}$$

where, by (46),

$$\phi_0(t) = a_{0,m}\left[-\frac{p^{2m}+q^m}{p^m(1+q^m)} + \cos mt\right] + b_{0,m}\sin mt$$

For the function $\phi_1(t)$, it follows from (45e), (45f) and (46) that

$$\phi_1(t) = a_{0,m}\left[-\cos\theta\frac{p^{2m}+q^m}{p^m(1+q^m)} + \cos\theta\cos mt\right] + b_{0,m}[-\cos\theta\sin mt].$$

Hence, the function ϕ can be written as

$$\phi(t) = a_{0,m}\varphi(t) + b_{0,m}\hat{\varphi}(t)$$

where φ and $\hat{\varphi}$ are given by (43); and $a_{0,m}$ and $b_{0,m}$ are arbitrary constants. This completes the proof of the theorem. □

The eigenfunctions corresponding to the eigenvalue $\lambda = -q^m$ can be obtained in a similar way.

5.2 Case $\cos\theta = 0$

For this case the eigenvalues are pure imaginary and are given by

$$\lambda = \pm q^n \mathrm{i}.$$

Theorem 5.2. *Suppose that* $\cos\theta = 0$ *and, for a fixed* m, $m = 1, 2, 3, \ldots$,

$$\lambda = q^m \mathrm{i}.$$

Then

$$\mathrm{Null}(\lambda\mathbf{I} - \mathbf{N}) = \mathrm{span}\{\varphi, \hat{\varphi}\}$$

where

$$\varphi(t) = \begin{cases} \alpha_1 + \cos mt, & t \in J_0, \\ -\mathrm{i}\sin\theta\sin mt, & t \in J_1, \end{cases} \qquad \hat{\varphi}(t) = \begin{cases} \sin mt, & t \in J_0, \\ \alpha_2\sin\theta - \mathrm{i}\sin\theta\cos mt, & t \in J_1, \end{cases} \tag{48}$$

and

$$\alpha_1 = -\frac{p^{2m}+\mathrm{i}q^m}{p^m(1+\mathrm{i}q^m)}, \quad \alpha_2 = -\frac{p^{2m}-\mathrm{i}q^m}{p^m(1+\mathrm{i}q^m)}. \tag{49}$$

Proof. The theorem is proved with the same arguments as in Theorem 5.1. □

The eigenfunctions corresponding to the eigenvalue $\lambda = -\mathrm{i}q^m$ can be obtained in a similar way.

5.3 Case $\sin\theta \neq 0$ and $\cos\theta \neq 0$

For this case the eigenvalues

$$\lambda = \pm q^n e^{\pm \mathrm{i}\theta}$$

are neither real nor pure imaginary.

Theorem 5.3. *Suppose that* $\sin\theta \neq 0$, $\cos\theta \neq 0$, *and, for a fixed* m, $m = 1, 2, 3, \ldots$,

$$\lambda = q^m e^{\mathrm{i}\theta}.$$

Then

$$\mathrm{Null}(\lambda\mathbf{I} - \mathbf{N}) = \mathrm{span}\{\varphi\}$$

where

$$\varphi(t) = \begin{cases} \alpha + \beta e^{\mathrm{i}\theta} + e^{\mathrm{i}mt}, & t \in J_0, \\ \beta + \alpha e^{\mathrm{i}\theta} + e^{-\mathrm{i}mt}, & t \in J_1, \end{cases} \tag{50}$$

and

$$\alpha = -\frac{p^{2m}}{p^m(1 + q^m e^{\mathrm{i}\theta})}, \quad \beta = -\frac{q^m}{p^m(1 + q^m e^{\mathrm{i}\theta})}. \tag{51}$$

Proof. In view of (42), Equations (27) become

$$\begin{aligned}
q^m e^{\mathrm{i}\theta} a_{0,0} &= -a_{0,0} - a_{0,m}p^m - [a_{1,m}\cos\theta - b_{1,m}\sin\theta](q/p)^m, & \text{(52a)}\\
e^{\mathrm{i}\theta} a_{0,m} &= a_{1,m}\cos\theta - b_{1,m}\sin\theta, & \text{(52b)}\\
e^{\mathrm{i}\theta} b_{0,m} &= -[a_{1,m}\sin\theta + b_{1,m}\cos\theta], & \text{(52c)}\\
q^m e^{\mathrm{i}\theta} a_{1,0} &= -a_{1,0} - a_{1,m}(q/p)^m - [a_{0,m}\cos\theta + b_{0,m}\sin\theta]p^m, & \text{(52d)}\\
e^{\mathrm{i}\theta} a_{1,m} &= a_{0,m}\cos\theta + b_{0,m}\sin\theta, & \text{(52e)}\\
e^{\mathrm{i}\theta} b_{1,m} &= a_{0,m}\sin\theta - b_{0,m}\cos\theta. & \text{(52f)}
\end{aligned}$$

Further, for $\lambda = q^m e^{\mathrm{i}\theta}$, it follows from (31) for $n = m$ that

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} a_{0,m} \\ b_{0,m} \end{pmatrix} = e^{2\mathrm{i}\theta} \begin{pmatrix} a_{0,m} \\ b_{0,m} \end{pmatrix} \tag{53}$$

which implies that

$$b_{0,m} = \mathrm{i}a_{0,m}. \tag{54}$$

Substituting (54) into (52e) and (52f) implies that

$$a_{1,m} = a_{0,m} \qquad b_{1,m} = -\mathrm{i}a_{0,m}. \tag{55}$$

Then, in view of (54) and (55), Equations (52b), (52c), (52e) and (52f) are satisfied. We substitute (54) and (55) into (52a) and choose the value of $a_{0,0}$ such that (52a) is satisfied, i.e.,

$$q^m e^{\mathrm{i}\theta} a_{0,0} = -a_{0,0} - a_{0,m}p^m - [a_{0,m}\cos\theta + \mathrm{i}a_{0,m}\sin\theta](q/p)^m,$$

and hence

$$a_{0,0} = a_{0,m}\left[-\frac{p^{2m} + q^m e^{\mathrm{i}\theta}}{p^m(1 + q^m e^{\mathrm{i}\theta})}\right] \tag{56}$$

Similarly, we substitute (54) and (55) into (52d) to obtain

$$q^m e^{\mathrm{i}\theta} a_{1,0} = -a_{1,0} - a_{0,m}(q/p)^m - [a_{0,m}\cos\theta + \mathrm{i}a_{0,m}\sin\theta]p^m,$$

and hence

$$a_{1,0} = a_{0,m}\left[-\frac{p^{2m}e^{\mathrm{i}\theta} + q^m}{p^m(1 + q^m e^{\mathrm{i}\theta})}\right] \tag{57}$$

Then, in view of Lemma 5.1, the eigenfunction ϕ corresponding to the eigenvalue $\lambda = \mathrm{i}q^m$ is given by

$$\phi(t) = \begin{cases} \phi_0(t), & t \in J_0, \\ \phi_1(t), & t \in J_1, \end{cases}$$

where, by (56) and (54),

$$\phi_0(t) = a_{0,m}\left[-\frac{p^{2m} + q^m e^{\mathrm{i}\theta}}{p^m(1 + q^m e^{\mathrm{i}\theta})} + \cos mt + \mathrm{i}\sin mt\right].$$

For the function $\phi_1(t)$, it follows from (55) and (57) that

$$\phi_1(t) = a_{0,m}\left[-\frac{p^{2m}e^{\mathrm{i}\theta} + q^m}{p^m(1 + q^m e^{\mathrm{i}\theta})} + \cos mt - \mathrm{i}\sin mt\right].$$

Hence, the function ϕ can be written as

$$\phi(t) = a_{0,m}\varphi(t)$$

where φ is given by (50) and $a_{0,m}$ is an arbitrary constant. This completes the proof of the theorem. □

The eigenfunctions corresponding to the eigenvalue $\lambda = -\mathrm{i}q^m e^{\mathrm{i}\theta}$ can be obtained in a similar way. Since $N(s,t)$ is real kernel, the eigenfunctions corresponding to the eigenvalues $\lambda = \mathrm{i}q^m e^{-\mathrm{i}\theta}$ and $\lambda = -\mathrm{i}q^m e^{-\mathrm{i}\theta}$ are the conjugate of the eigenfunctions corresponding to the eigenvalues $\lambda = \mathrm{i}q^m e^{\mathrm{i}\theta}$ and $\lambda = -\mathrm{i}q^m e^{\mathrm{i}\theta}$, respectively.

6. Final remarks

The boundary integral equation with the generalized Neumann kernel $N(s,t)$ formed with the function

$$A(t) = e^{\mathrm{i}\nu(t)}(\eta(t) - p),$$

has many applications to several problems in multiply connected domains in the complex plane such as numerical computing of conformal mappings, numerical solution of boundary value problems, and numerical computing of ideal fluid flow (see Nasser (2015b) for more details). The integral equation can be solved numerically by discretizing it by the Nyström method with the trapezoidal rule to obtain a dense and nonsymmetric linear system. The discretized linear system is solved by GMRES method combined with FMM. The GMRES method requires only few iterations to converge to a very good approximation of the exact solution, see Nasser (2015a,b), Nasser and Al-Shihri (2013), Nasser and Green (2018), Nasser et al. (2016, 2015).

For better understanding of the very fast convergence of GMRES method, several numerical experiments have been done to study the properties of the eigenvalues of the kernel $N(s,t)$ and these numerical results confirm that, except for -1, the the eigenvalues of $N(s,t)$ are in the unit disk and clustered around 0 (and hence the eigenvalues of the matrix of discretized linear system are strongly clustered around 1), see Nasser (2015a,b), Nasser and Al-Shihri (2013), Nasser et al. (2011). However, no analytical proof for such properties of the eigenvalues is yet available.

In this paper, we looked in close detail at the eigenvalues of the generalized Neumann kernel $N(s,t)$ for the annular domain $q < |z| < 1$. Without lost of generality, we assumed

$$A(t) := \begin{cases} e^{\mathrm{i}t} - p, & t \in J_0, \\ e^{\mathrm{i}\theta}(qe^{-\mathrm{i}t} - p), & t \in J_1, \end{cases}$$

where θ and p are real numbers with $q < p < 1$. We proved that the eigenvalues of $N(s,t)$ are

$$-1 \quad \text{and} \quad \pm\, q^m e^{\pm \mathrm{i}\theta}, \quad m = 1, 2, 3, \ldots.$$

Except of -1, these eigenvalues are indeed in the unit disk and strongly clustered around 0. Thus, the analytic results presented in this paper for the annular domain agreed with the numerical results obtained earlier in Nasser (2015a,b), Nasser and Al-Shihri (2013), Nasser et al. (2011) for general multiply connected domains. However, proving such an important property of the eigenvalues of the generalized Neumann kernel for general multiply connected domains is still an open problem.

When $A = 1$, the kernel N in (6) is the well-known Neumann kernel which appears frequently in integral equations in potential theory and conformal mappings, see Henrici (1993), Kress (2014), Kythe (2012), Kythe and Puri (2011). The eigenvalues of the Neumann kernel for bounded multiply connected domains are real in the interval $[-1, 1]$, Nasser et al. (2011). For simply connected domains bounded by ellipses, the explicit forms of the eigenvalues of the Neumann kernel have been presented in Muminov and Murid (2018).

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